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1987 J. Phys. A: Math. Gen. 20 1565

(http://iopscience.iop.org/0305-4470/20/6/038)

J. Phys. A: Math. Gen. 20 (1987) 1565-1585. Printed in the UK

Exact solution of the integrable XXZ Heisenberg model with arbitrary spin: I. The ground state and the excitation spectrum

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Received 4 March 1986

Abstract. An integrable generalisation of the XXZ Heisenberg model with arbitrary spin and with light plane type anisotropy is studied. Integral equations describing the thermodynamics of the model are found. The antiferromagnetic ground state, the excitation spectrum, the quantum numbers and scattering amplitudes of the excitations are determined.

1. Introduction

The Heisenberg-Ising model describes a chain of atoms with nearest-neighbour interaction. The Hamiltonian of the model with anisotropy of the XXZ type has the following form:

$$\mathcal{H} = \sum_{n=1}^{N} (\sigma_n^1\sigma_{n+1}^1 + \sigma_n^2\sigma_{n+1}^2 + \Delta\sigma_n^3\sigma_{n+1}^3). \tag{1.1}$$

Here σ_n^α are local spin operators (Pauli's matrices). The light plane type anisotropy corresponds to $|\Delta| < 1$. In this case it is more convenient to use the parameter γ, where $\cos\gamma = \Delta$.

Since the pioneering paper of Bethe (1931) the quantum Heisenberg-Ising chain of spin $\frac{1}{2}$ has been studied by many authors. Among them Yang and Yang (1966) and des Cloizeaux and Gaudin (1966) computed the ground-state energy and the dispersion law of the excitations. The thermodynamics of the model were described by Takahashi and Suzuki (1972). Correct quantum numbers of the excitations and their scattering amplitudes were found by Babelon *et al* (1983).

The development of the quantum inverse transform method (QITM) (see Faddeev and Takhtajan 1979, Kulish and Sklyanin 1982) has led to the important notion of the R matrix and established its crucial role in the theory of quantum integrable systems. These matrices describe commutation relations between the elements of the quantum monodromy matrix. QITM opens the way for a systematic construction of the families of integrable systems connected with given R matrices. In particular, the XXX model of higher spin (Kulish and Sklyanin 1982, Takhtajan 1982, Babujian 1982), the lattice version of the sine-Gordon model and some other models were constructed along these lines (Izergin and Korepin 1982).

The present paper is concerned with the exact solution of an integrable generalisation of the model (1) to the case of arbitrary spin. In the case of anisotropy of the light axis type the ground state and the excitation spectrum were determined by Sogo (1984).

This model is formulated in the papers by Kulish and Reshetikhin (1981) and Kulish and Sklyanin (1982).

2. Description of the model

In order to generalise the model (1) to the case of arbitrary spin we start with the description of the corresponding R matrix.

Let $V_i \simeq \mathbb{C}^{l_i+1}$ be the irreducible SU(2) module with spin $l_i/2$. By definition, $R_{ij}^{(l_i,l_j)}(u)$ is a linear operator acting in $V_i \otimes V_j$ which depends on a complex parameter u and satisfies the Yang–Baxter equation

$$R_{12}^{(l_1,l_2)}(u)R_{13}^{(l_1,l_3)}(u+v)R_{23}^{(l_2,l_3)}(v)=R_{23}^{(l_2,l_3)}(v)R_{13}^{(l_1,l_3)}(u+v)R_{12}^{(l_1,l_2)}(u). \tag{2.1}$$

The simplest R matrix is that corresponding to the spin-$\frac{1}{2}$ representation both in V_i and V_2. It is acting in $\mathbb{C}^2\otimes\mathbb{C}^2$ and is given by

$$R^{(1,1)}(u)=\sinh\left(u+\mathrm{i}\gamma\frac{1+\sigma^3\otimes\sigma^3}{2}\right)+\mathrm{i}\sin\gamma(\sigma^+\otimes\sigma^-+\sigma^-\otimes\sigma^+) \tag{2.2}$$

where $\sigma^\pm=\sigma^1\pm\mathrm{i}\sigma^2$, σ^α are the Pauli matrices. Using the fusion procedure for R matrices (see Kulish *et al* 1981), it is not difficult to construct the $R^{(l,m)}(u)$, acting in $\mathbb{C}^{l+1}\otimes\mathbb{C}^{m+1}$:

$$\begin{aligned}R_{(a)(b)}^{(l,m)}(u)=&(A^{(l)}\otimes A^{(m)})(P_{(a)}^+\otimes P_{(b)}^+)\prod_{k=1}^{l}\prod_{j=1}^{m}R_{a_kb_j}^{(1,1)}[u+\mathrm{i}\gamma(l-m-k-j)]\\&\times(P_{(a)}^+\otimes P_{(b)}^+)(A^{(l)^{-1}}\otimes A^{(m)^{-1}})g_{lm}^{-1}(u).\end{aligned} \tag{2.3}$$

Here

$$g_{lm}(u)=\prod_{k=1}^{l-1}\prod_{j=0}^{m-1}\sinh[u+\mathrm{i}\gamma(l+m-k-j+1)] \qquad l\geqslant m \tag{2.4}$$

and $P_{(a)}^+$, $P_{(b)}^+$ are projection operators onto the subspaces of completely symmetric tensors in $V_{a_1}\otimes\ldots\otimes V_{a_l}$ and $V_{b_1}\otimes\ldots\otimes V_{b_m}$, respectively, with $V_{a_i}\simeq V_{b_i}\simeq\mathbb{C}^2$. The product in (2.3) is ordered in accordance with the increase of indices. The matrix $A^{(l)}$ is defined in the following way:

$$A^{(l)}=\sum_a |R_a|^{1/2}\mathcal{P}^{(a)} \tag{2.5}$$

where R_a and $\mathcal{P}^{(a)}$ are defined by the spectral decomposition

$$\prod_{k<j}R_{a_ka_j}[\mathrm{i}\gamma(k-j)]=\sum_a R_a\mathcal{P}^{(a)}. \tag{2.6}$$

The matrix (2.3) is analytic, without zeros in the whole complex plane of variable u. It has simple asymptotics at $u\to\infty$

$$\begin{aligned}R^{(l,m)}(u)\simeq&\left(\frac{1}{2}\right)^m\exp\left(mu+\mathrm{i}\frac{lm}{2}\gamma+\mathrm{i}\frac{\gamma}{2}S^3\otimes S^3\right)\\&\times\left\{1+\mathrm{e}^{-u}\left[S^-\exp\left(\mathrm{i}\frac{\gamma}{2}S^3\right)\otimes\exp\left(\mathrm{i}\frac{\gamma}{2}S^3\right)S^+\right.\right.\\&\left.\left.+\exp\left(\mathrm{i}\frac{\gamma}{2}S^3\right)S^+\otimes S^-\exp\left(-\mathrm{i}\frac{\gamma}{2}S^3\right)\right]+\mathrm{O}(\mathrm{e}^{-2u})\right\} \qquad u\to-\infty\end{aligned} \tag{2.7}$$

where the operators $S^{\pm}$, S^3 satisfy the following commutation relations:

$$[S^3, S^{\pm}] = \pm 2S^{\pm} \qquad [S^+, S^-] = \sin(\gamma S^3)/\sin\gamma. \tag{2.8}$$

Let $\mathscr{A}$ be the free associative algebra generated by the elements $S^{\pm}$, S^3 satisfying the commutations (2.8). The space $\mathbb{C}^{l+1}$ is a natural module over $\mathscr{A}$ with the highest weight vector $|0\rangle$

$$S^3|0\rangle = l|0\rangle \qquad S^+|0\rangle = 0. \tag{2.9}$$

Note that for $m = 1$ the matrix (2.3) may be easily expressed through the elements of the algebra (2.8)

$$R^{(l,1)}(u) = \sinh\left(u + \mathrm{i}\gamma\frac{l + S^3 \otimes \sigma^3}{2}\right) + \mathrm{i}\sin\gamma(S^+ \otimes \sigma^- + S^- \otimes \sigma^+). \tag{2.10}$$

Substituting (2.10) into (2.3) yields a simplified expression

$$R^{(l,m)}_{a(b)}(u) = A^{(m)}_{(b)} P^+_{(b)} \prod_{j=1}^{m} R^{(l,1)}_{ab_j}[u + \mathrm{i}\gamma(j-1)] P^+_{(b)} A^{(m)^{-1}}_{(b)} \qquad l \geqslant m. \tag{2.11}$$

The spectral decomposition of (2.3) was obtained by Jimbo (1985).

At $u = \pm \mathrm{i}\gamma$ the matrix (2.10) degenerates to

$$R^{(l,1)}_{12}(\mathrm{i}\gamma) = B^{(l)}_{12} \mathscr{P}^{(l+1)}_{12} \qquad R^{(l,1)}_{12}(-\mathrm{i}\gamma) = C^{(l)}_{12} \mathscr{P}^{(l-1)}_{12}. \tag{2.12}$$

Here $\mathscr{P}^{(l\pm1)}$ are the projection operators onto the subspaces of the spin $(l \pm 1)/2$ in the decomposition of the tensor product of the $\mathscr{A}$ moduli of spin $l/2$ and spin $\frac{1}{2}$ (rank $\mathscr{P}^{(l+1)} = l+2$, rank $\mathscr{P}^{(l-1)} = l$).

Using (2.1), the properties (2.12) and the fusion procedure of Kulish *et al* (1981), one can prove that the following relations hold:

$$D^{(l)}_{12} R^{(l,m)}_{13}(u + \mathrm{i}\gamma) R^{(1,m)}_{23}(u) D^{(l)^{-1}}_{12} = \begin{pmatrix} \sinh(u + \mathrm{i}\gamma m) R^{(l+1,m)}_{(12),3}(u) & 0 \\ * & \sinh u\, R^{(l-1,m)}_{\langle 12\rangle,3}(u + 2\mathrm{i}\gamma) \end{pmatrix} \qquad l > m \tag{2.13}$$

$$D^{(m)}_{12} R^{(m,m)}_{13}(u + \mathrm{i}\gamma) R^{(1,m)}_{23}(u) D^{(m)^{-1}}_{12} = \begin{pmatrix} \sinh(u + \mathrm{i}\gamma m) R^{(m+1,m)}_{(12),3}(u) & 0 \\ * & \sinh u \sinh[u + \mathrm{i}\gamma(m+1)] R^{(m-1,m)}_{\langle 12\rangle,3}(u + 2\mathrm{i}\gamma) \end{pmatrix} \tag{2.14}$$

$$D^{(l)}_{12} R^{(l,m)}_{13}(u + \mathrm{i}\gamma) R^{(1,m)}_{23}(u) D^{(l)^{-1}}_{12} = \begin{pmatrix} R^{(l+1,m)}_{(12),3}(u) & 0 \\ * & \sinh u \sinh[u + \mathrm{i}\gamma(m+1)] R^{(l-1,m)}_{\langle 12\rangle,3}(u + 2\mathrm{i}\gamma) \end{pmatrix} \qquad l < m. \tag{2.15}$$

Here the block structure is in accordance with the decomposition (2.13) of the tensor product $V_1 \otimes V_2$, $V_{\langle 12\rangle} \simeq \mathbb{C}^l$, $V_{(12)} \simeq \mathbb{C}^{l+2}$.

The definition of $D^{(l)}$ is similar to that of $A^{(l)}$ from (2.3), namely

$$\mathcal{P}^{(l+1)}B^{(l)}\mathcal{P}^{(l+1)}=\sum_a B_a^{(l)}\mathcal{P}^{(a)},\ \mathcal{P}^{(l-1)}C^{(l)}\mathcal{P}^{(l-1)}=\sum_a C_a^{(l)}\mathcal{P}^{(a)} \tag{2.16}$$

$$D^{(l)}=\begin{pmatrix} \sum_a |B_a^{(l)}|^{1/2}\mathcal{P}^{(a)} & 0 \\ 0 & \sum_a |C_a^{(l)}|^{1/2}\mathcal{P}^{(a)} \end{pmatrix} \tag{2.17}$$

where $\mathcal{P}^{(a)}$ are the projection operators onto the corresponding eigensubspaces of matrices $B^{(l)}$ and $C^{(l)}$ from (2.12).

In order to describe the integrable generalisation of the Heisenberg model (1.1) to the case of higher spin, let us define the family of the monodromy matrices

$$T^{(l)}(u)=R_{a1}^{(l,m)}(u)\ldots R_{aN}^{(l,m)}(u). \tag{2.18}$$

These matrices act in the tensor product $V_a\otimes\mathscr{h}$ where $\mathscr{h}=V_1\otimes\ldots\otimes V_N$ is the quantum space of the system and V_a is the auxiliary space, $V_i\simeq\mathbb{C}^m$, $V_a\simeq\mathbb{C}^l$.

By definition, the relative trace of the monodromy matrix over the auxiliary space gives the transfer matrix. Transfer matrices form a commutative family of operators acting in $\mathscr{h}$ (Baxter 1972, Faddeev and Takhtajan 1979, Kulish and Sklyanin 1982)

$$t_l^{(m)}(u)=\mathrm{Tr}_a T^{(l)}(u) \qquad [t_l^{(m)}(u),t_n^{(m)}(v)]=0. \tag{2.19}$$

From (2.14)-(2.16) we obtain the recurrence relations for the transfer matrices (2.18)

$l>m$

$$t_1^{(m)}(u)t_l^{(m)}(u+\mathrm{i}\gamma)=\sinh(u+\mathrm{i}\gamma m)^N t_{l+1}^{(m)}(u)+\sinh u^N\, t_{l-1}^{(m)}(u+2\mathrm{i}\gamma) \tag{2.20}$$

$$t_1^{(m)}(u)t_m^{(m)}(u+\mathrm{i}\gamma)=\sinh(u+\mathrm{i}\gamma m)^N t_{m+1}^{(m)}(u) \\ +\sinh u^N\sinh[u+\mathrm{i}\gamma(m+1)]^N t_{m-1}^{(m)}(u+2\mathrm{i}\gamma) \tag{2.21}$$

$l<m$

$$t_1^{(m)}(u)t_l^{(m)}(u+\mathrm{i}\gamma)=t_{l+1}^{(m)}(u)+\sinh u^N\sinh[u+\mathrm{i}\gamma(m+1)]^N t_{l-1}^{(m)}(u+2\mathrm{i}\gamma). \tag{2.22}$$

Using these relations one can express all $t_l^{(m)}(u)$ through $t_1^{(m)}(u)$ algebraically. The easiest way to get such an expression is by means of the generating function for $t_l^{(m)}(u)$

$$(1-zt_1^{(m)}(u)+z^2d^{(m)}(u))^{-1} \\ =\sum_{k=0}^{m} z^k t_k^{(m)}(u)+\sum_{k=m+1}^{\infty} z^k\prod_{l=m}^{k-1}\sinh(u+\mathrm{i}\gamma l)^N t_k^{(m)}(u) \tag{2.23}$$

where z is the shift operator

$$z^{-1}f(u)z=f(u+\mathrm{i}\gamma) \tag{2.24}$$

and $d^{(m)}(u)$ is the quantum determinant of the matrix $T^{(1)}(u)$ (Kulish and Sklyanin 1982)

$$d^{(m)}(u)=\sinh u^N\sinh[u+\mathrm{i}\gamma(m+1)]^N. \tag{2.25}$$

The eigenvalues of $t_1^{(m)}(u)$ are easily calculated by means of the algebraic Bethe ansatz (Faddeev and Takhtajan 1979). They are characterised by the set of the numbers $\{u_j\}_1^n$, $n\leqslant mN/2$, and have the following form

$$\Lambda_1^{(m)}(u)=\sinh(u+\mathrm{i}\gamma m)^N\prod_{j=1}^{n}\frac{\sinh(u-u_j-\mathrm{i}\gamma)}{\sinh(u-u_j)}+\sinh u^N\prod_{j=1}^{n}\frac{\sinh(u-u_j+\mathrm{i}\gamma)}{\sinh(u-u_j)}. \tag{2.26}$$

The numbers u_j are the solutions of the system of Bethe equations. It is more convenient to write this system in terms of the new variables $\lambda_j = 2u_j/\gamma + \mathrm{i}m$

$$\left(\frac{\sinh[\frac{1}{2}\gamma(\lambda_j+\mathrm{i}m)]}{\sinh[\frac{1}{2}\gamma(\lambda_j-\mathrm{i}m)]}\right)^N = \prod_{k\neq j}^{n} \frac{\sinh[\frac{1}{2}\gamma(\lambda_j-\lambda_k+2\mathrm{i})]}{\sinh[\frac{1}{2}\gamma(\lambda_j-\lambda_k-2\mathrm{i})]}. \tag{2.27}$$

Substituting (2.26) into the generating function (2.23) we obtain an exact expression for the eigenvalues of $t_l^{(m)}(u)$ in terms of numbers u_j

$$\begin{aligned}\Lambda_l^{(m)}(u) = &\sum_{j=1}^{l-1} a_j(u)^N \prod_{k=1}^{n} \frac{\sinh(u-u_k+\mathrm{i}\gamma l)\sinh(u-u_k-\mathrm{i}\gamma)}{\sinh(u-u_k+\mathrm{i}\gamma j)\sinh[u-u_k+\mathrm{i}\gamma(j-1)]}\\ &+ a_l(u)^N \prod_{k=1}^{n} \frac{\sinh(u-u_k-\mathrm{i}\gamma)}{\sinh[u-u_k+\mathrm{i}\gamma(l-1)]} + a_0(u)^N \prod_{k=1}^{n} \frac{\sinh(u-u_k+\mathrm{i}\gamma l)}{\sinh(u-u_k)}\end{aligned} \tag{2.28}$$

where

$$a_j(u) = \prod_{q=j}^{l-1} \sinh(u+\mathrm{i}\gamma q) \prod_{p=0}^{j-1} \sinh[u+\mathrm{i}\gamma(m+p)]. \tag{2.29}$$

Before describing the Hamiltonian of the higher spin *XXZ* model and its spectrum let us make some comments on the formula (2.28) for the eigenvalues of $t_l^{(m)}(u)$.

The equations (2.20)-(2.22) generate a system of functional equations for $\Lambda_l^{(m)}(u)$. These equations together with the crossing-symmetry relations

$$t_l^{(m)}(u)^t = (-1)^{\min(l,m)} t_l^{(m)}[-u-\mathrm{i}\gamma(l+m-1)] \tag{2.30}$$

(where t is a transposition) have only a finite number of solutions in the class of Lourent polynomials of degree lN in the variable e^u. The expressions (2.28) together with the system (2.27) give the solutions of this system. Equations (2.27) are the conditions that all residues of $\Lambda_l^{(m)}(u)$ at finite u be equal to zero. This interpretation of the formulae (2.27) and (2.28) gives an alternative approach to the calculation of the eigenvalues of transfer matrices (Reshetikhin 1983) (the so-called analytical Bethe ansatz). The idea of this method goes back to Baxter's work (Baxter 1972).

The local Hamiltonian with nearest-neighbour interactions is a logarithmic derivative of $t_m^{(m)}(u)$ at $u=-\mathrm{i}\gamma m$:

$$\begin{aligned}\mathscr{H} &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}u}\log t_m^{(m)}(u)\Big|_{u=-\mathrm{i}\gamma m} - Na_m'(-\mathrm{i}\gamma m)\\ &= -\mathrm{i}\sum_{n=1}^{N} P_{nn+1}\frac{\mathrm{d}}{\mathrm{d}u} R_{nn+1}^{(m,m)}(u)\Big|_{u=-\mathrm{i}\gamma m} - Na_m'(-\mathrm{i}\gamma m).\end{aligned} \tag{2.31}$$

It is a generalisation of the Hamiltonian (1.1) to the case of arbitrary spin. An exact expression for this Hamiltonian in terms of spin operators was obtained by Fateev and Zamolodchikov (1980) for $S=m/2=1$ and by Jimbo (1985) for the case of arbitrary spin.

It is not difficult to see that the operator (2.31) commutes with the operator of spin projection

$$[\mathscr{H}, S_z] = 0 \qquad S_z = \sum_{n=1}^{N} S_n^3 \tag{2.32}$$

where S_n^3 are defined by (2.8). The operator S_z is diagonal in the eigenbasis of the transfer matrix:

$$S_z|u_1, \ldots, u_n\rangle = (\tfrac{1}{2}mN - n)|u_1, \ldots, u_n\rangle. \tag{2.33}$$

The Hamiltonian (2.31) changes sign under the replacement $\gamma \to \pi - \gamma$:

$$\mathcal{H}(\gamma) = -J\mathcal{H}(\pi - \gamma)J \tag{2.34}$$

where

$$J = \exp\left(\mathrm{i}\pi \sum_{n \text{ even}} S_n^3\right).$$

Therefore, instead of considering $\mathcal{H}(\gamma)$ in the interval $0 < \gamma < \pi$, we will investigate separately the ground state and excitations of $\mathcal{H}(\gamma)$ and $-\mathcal{H}(\gamma)$ at $0 < \gamma < \pi/2$.

Adding the interaction with the magnetic field H, we obtain a final expression for the Hamiltonian of the model under consideration:

$$\mathcal{H}(H, \gamma) = \varepsilon\mathcal{H}(\gamma) - S_z H \tag{2.35}$$

where $\varepsilon = \pm 1$, $0 < \gamma < \pi/2$, $H > 0$.

The spectrum of this Hamiltonian may be calculated from (2.28) and (2.31):

$$E = \sum_{k=1}^{n} \left(-\frac{\varepsilon \sin(m\gamma)}{\sinh[\frac{1}{2}\gamma(\lambda_k + \mathrm{i}m)] \sinh[\frac{1}{2}\gamma(\lambda_k - \mathrm{i}m)]} + H \right) - \frac{mN}{2} H \tag{2.36}$$

where the numbers λ_k are the solutions of the system (2.27).

The full momentum P is defined as the logarithm of the translation operator:

$$t_m^{(m)}(-\mathrm{i}\gamma m) = \text{constant} \times p_{12} \ldots p_{1N} \equiv \text{constant} \times \exp(\mathrm{i}P) \tag{2.37}$$

$$\mathrm{e}^{-\mathrm{i}P}\mathcal{O}_n \mathrm{e}^{\mathrm{i}P} = \mathcal{O}_{n+1}. \tag{2.38}$$

The eigenvalues of P may be found from (2.28) and (2.31):

$$P = \sum_{k=1}^{n} 2\tan^{-1}\left(\tanh\left(\frac{\gamma\lambda}{2}\right)\cot\left(\frac{\gamma m}{2}\right)\right). \tag{2.39}$$

Now we have finished the algebraic part. In the next section we will consider our model in the thermodynamical limit.

3. The thermodynamic equations

In the thermodynamical limit n/N is fixed, $0 \leq n/N < S$ and $N \to \infty$. In this limit the solutions of the system (2.27) may be arranged in the so-called strings. A string of length n and parity $v_n = \pm 1$ is defined as a family of $n\lambda_\alpha$ with the same real part, of the form

$$\lambda_{\alpha j}^n = \lambda_\alpha^n + \mathrm{i}[n + 1 - 2j + \tfrac{1}{2}(1 - v_m v_n)p_0] + \mathrm{O}(\mathrm{e}^{-\delta N}) \qquad \delta > 0. \tag{3.1}$$

The number λ_α^n is called the string's centre and $v_m = \pm 1$ is the spin parity (see (3.8)), $m = 2S$.

To describe admissible lengths and parities of strings let us define the set of numbers p_i and the sets of the numbers b_i, y_i, m_i:

$$p_0 = \pi/\gamma \qquad p_1 = 1 \qquad b_i = [p_i/p_{i+1}] \qquad p_{i+1} = p_{i-1} - b_{i-1}p_i \qquad i \geqslant 1 \tag{3.2}$$

$$y_{-1} = 0 \qquad y_0 = 1 \qquad y_1 = b_0 \qquad y_{i+1} = y_{i-1} + b_i y_i \qquad i \geqslant 0 \tag{3.3}$$

$$m_0 = 0 \qquad m_1 = b_0 \qquad m_{i+1} = m_i + b_i \qquad i \geqslant 0. \tag{3.4}$$

The numbers b_i are defined by the decomposition of p_0 into the continued fraction

$$p_0 = [b_0, b_1, b_2, \ldots] = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2 + \ldots}} \tag{3.5}$$

$$p_i/p_{i+1} = [b_i, b_{i+1}, \ldots]. \tag{3.6}$$

The following sequences were first introduced by Takahashi and Suzuki (1972):

$$n_j = y_{i-1} + (j - m_i)y_i \qquad m_i \leqslant j < m_{i+1} \tag{3.7}$$

$$v_j \equiv v_{n_j} = \exp\left(\mathrm{i}\pi\left[\frac{n_j - 1}{p_0}\right]\right) \qquad j \neq m_1,\ v_{m_1} = -1. \tag{3.8}$$

For $S = \frac{1}{2}$ one can prove that in the states with any $0 \leqslant n/N < S$ the lengths and the parities of the strings are given by (3.7) and (3.8). There are no restrictions on the position of the centres of strings in this case. If $S > \frac{1}{2}$ the lengths and the parities of the admissible strings also form the series (3.7) and (3.8). But now the restrictions on the centres of the strings may appear. There are no such restrictions however for all states with $0 \leqslant n/N < S$ if $1 + 2S$ is the Takahashi number (Kirillov and Reshetikhin 1985a,b)

$$1 + 2S = n_\sigma \qquad m_r \leqslant \sigma < m_{r+1}. \tag{3.9}$$

Here n_σ is one of the numbers (3.7).

If p_0 is a rational number, $p_{\alpha+2} = 0$ and $i \leqslant \alpha$ in (3.2)-(3.4). In this case $j \leqslant m_{\alpha+1}$ and $n_j < y_{\alpha+1} = u$, where u is the nominator of the fraction $p_0 = u/v$. Next we shall use also the set of numbers q_j:

$$q_j = (-1)^i[p_i - (j - m_i)p_{i+1}] \qquad m_i \leqslant j < m_{i+1}. \tag{3.10}$$

To obtain a system of equations for the centres of strings we substitute (3.1) in (3.7) and consider the product of these equations for $\lambda^{n}_{\alpha j}$ over $1 \leqslant j \leqslant n$. Taking logarithms we obtain the system $(\lambda^{(j)}_\alpha \equiv \lambda^{n_j}_\alpha)$

$$N t_{j,2S}(\lambda^{(j)}_\alpha) = 2\pi I^j_\alpha + \sum_{k \geqslant 1} \sum_{\beta=1}^{\nu_k} \Theta_{jk}(\lambda^{(j)}_\alpha - \lambda^{(k)}_\beta). \tag{3.11}$$

Here ν_j is the number of j strings. I^j_α is an integer (or a half-integer). The functions $t_{j,2S}(\lambda)$ and $\Theta_{jk}(\lambda)$ were calculated directly from (2.27) and have the form

$$t_{j,2S}(\lambda) = \sum_{l=1}^{\min(n_j, 2S)} f(\lambda, |n_j - 2S| + 2l - 1, v_j v_\sigma) \tag{3.12}$$

$$\Theta_{jk}(\lambda) = f(\lambda, |n_j - n_k|, v_j v_k) + f(\lambda, n_j + n_k, v_j v_k) + 2 \sum_{l=1}^{\min(n_j, n_k) - 1} f(\lambda, |n_j - n_k| + 2l, v_j v_k) \tag{3.13}$$

$$f(\lambda, n, v) = \begin{cases} 2\tan^{-1}\left\{\tan\left[\left(\dfrac{1+v}{4} - \dfrac{n}{2p_0}\right)\pi\right]\tanh\left(\dfrac{\pi\lambda}{2p_0}\right)\right\} & \text{if } \dfrac{n}{p_0} \text{ is non-integer} \\ 0 & \text{if } \dfrac{n}{p_0} \text{ is integer.} \end{cases} \tag{3.14}$$

We use the notation of Takahashi and Suzuki (1972).

In the thermodynamic limit the centres of the j strings fill the real axis with the densities $\rho_j(\lambda)$. Following Yang and Yang (1966) we introduce the hole densities $\rho_j^{\mathrm{h}}(\lambda)$. The value $\rho_j^{\mathrm{h}}(\lambda)\,\mathrm{d}\lambda$ is the number of j strings which are absent in the interval $\mathrm{d}\lambda$ in a given state.

The state (2.26), which at finite N is characterised by the numbers I_α^j at $N \to \infty$, is described by the densities $\rho_j(\lambda)$. For the densities $\rho_j(\lambda)$ and $\rho_j^{\mathrm{h}}(\lambda)$ from (3.11) we obtain a system of integral equations:

$$a_{j,2S}(\lambda) = (-1)^{r(j)}\rho_j^{\mathrm{h}}(\lambda) + \sum_{k\geqslant 1} A_{jk} * \rho_k(\lambda) \tag{3.15}$$

where $r(j) = i$ if $m_i \leqslant j < m_{i+1}$ and

$$\begin{aligned} a_{j,2S}(\lambda) &= \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda} t_{j,2S}(\lambda) \\ A_{jk}(\lambda) &= \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\lambda}\Theta_{jk}(\lambda) + (-1)^{r(j)}\delta_{jk}\delta(\lambda). \end{aligned} \tag{3.16}$$

Here $a*b$ is the convolution of functions $a(\lambda)$, $b(\lambda)$ defined by

$$(a*b)(\lambda) = \int_{-\infty}^{+\infty} a(\lambda-\mu)b(\mu)\,\mathrm{d}\mu.$$

From (3.12)-(3.14) one can obtain the expressions for the functions $a_{j,2S}(\lambda)$ and $A_{jk}(\lambda)$. To describe it let us introduce the functions $a_j(\lambda)$ and $n_j(\lambda)$ defined by their Fourier transforms

$$\hat{n}_j(x) = \cosh\left[\left(\left\{\frac{n_j}{p_0}\right\} - \frac{1-(-1)^{r(j)}}{2}\right)p_0 x\right] + \sum_{l=1}^{n_j-1}\cosh\left[\left(\left\{\frac{n_j - l}{p_0}\right\} - \left\{\frac{l}{p_0}\right\}\right)p_0 x\right] \tag{3.17}$$

$$\hat{a}_j(x) = \frac{\sinh(q_j x)}{\sinh(p_0 x)}. \tag{3.18}$$

Here $\{x\}$ is the fractional part of x.

We use the following normalisation of the Fourier transform:

$$\hat{f}(x) = \int_{-\infty}^{+\infty} \exp(-\mathrm{i}\lambda x) f(\lambda)\,\mathrm{d}\lambda \qquad f(\lambda) = \int_{-\infty}^{+\infty}\exp(\mathrm{i}\lambda x)\hat{f}(x)\frac{\mathrm{d}x}{2\pi}. \tag{3.19}$$

The following statements follow from direct calculations.

Proposition 1. Let $k \geqslant j$, then

$$\hat{A}_{kj}(x) = \hat{A}_{jk}(x) = 2\hat{a}_k(x)\hat{n}_j(x) + (-1)^{r(k)}\delta_{k,m_{\alpha+1}}\delta_{j,m_{\alpha+1}-1}. \tag{3.20}$$

Proposition 2. Let $m_r < \sigma < m_{r+1}$, then

$$\hat{a}_{j,2S}(x) = \hat{A}_{j,\sigma-1}(x)\hat{S}_{r+1}(x) + 2\cosh(q_\sigma x)\sum_{l=1}^{r}\hat{A}_{j,m_l-1}(x)\hat{S}_l(x)\hat{S}_{l+1}(x) \tag{3.21}$$

$$\hat{S}_i(x) = 1/2\cosh(p_i x).$$

If $\sigma = m_r$, the first term in (3.21) is absent.

In appendix 1 some helpful identities are given for the functions $\hat{a}_j(x)$ and $\hat{n}_j(x)$.

Passing to the limit $N\to\infty$ in equation (2.36) we obtain the energy of the state corresponding to the set of densities $\rho_j(\lambda)$:

$$\mathscr{E}=\frac{E}{N}=\sum_{j\geq 1}\int_{-\infty}^{+\infty}\left(\frac{4\pi}{\gamma}a_{j,2S}(\lambda)+n_jH\right)\rho_j(\lambda)\,d\lambda-SH. \tag{3.22}$$

We obtain the equilibrium state by minimisation of the free energy functional

$$F=\mathscr{E}-TS \tag{3.23}$$

where S is the combinatorial entropy

$$S=\sum_{j\geq 1}\int_{-\infty}^{+\infty}[(\rho_j(\lambda)+\rho_j^{\mathrm{h}}(\lambda))\log(\rho_j(\lambda)+\rho_j^{\mathrm{h}}(\lambda))-\rho_j(\lambda)\log\rho_j(\lambda)-\rho_j^{\mathrm{h}}(\lambda)\log\rho_j^{\mathrm{h}}(\lambda)]\,d\lambda. \tag{3.24}$$

Using equations (3.15) one can express the free energy of our model in terms of the solution of the system of non-linear integral equations

$$\varepsilon\frac{4\pi}{\gamma}a_{j,2S}(\lambda)+n_jH-T\log[1+\exp(\beta\varepsilon_j)]$$
$$+\sum_{k\geq 1}(-1)^{r(k)}A_{jk}*T\log[1+\exp(-\beta\varepsilon_k)]=0 \tag{3.25}$$

$$F(H,T)=-\sum_{j\geq 1}\int_{-\infty}^{+\infty}(-1)^{r(j)}a_{j,2S}(\lambda)T\log[1+\exp(-\beta\varepsilon_j(\lambda))]\,d\lambda. \tag{3.26}$$

Here the functions $\varepsilon_j(\lambda)$ are the energies of j strings and $\exp(-\beta\varepsilon_j(\lambda))=\rho_j(\lambda)/\rho_j^{\mathrm{h}}(\lambda)$, $\beta=1/T$ and T is the temperature.

Equations (3.25) and the expression (3.26) completely determine the thermodynamical properties of the model.

4. The ground state of the model and the excitations

Now let us study the ground state and the excitation spectrum at $T\to 0$. In this case equations (3.25) becomes linear

$$\varepsilon(4\pi/\gamma)a_{j,2S}+n_jH-\varepsilon_j^{+}-\sum_{k\geq 1}(-1)^{r(k)}A_{jk}*\varepsilon_k^{+}=0. \tag{4.1}$$

Here $\varepsilon_j^{\mp}(\lambda)$ are the negative and positive parts of the functions $\varepsilon_j(\lambda)$. An expression for the ground-state energy may be obtained from (3.26) in the limit

$$\mathscr{E}_0(H)=F(H,0)=\sum_{j\geq 1}(-1)^{r(j)}\int_{-\infty}^{+\infty}a_{j,2S}(\lambda)\varepsilon_j^{-}(\lambda)\,d\lambda. \tag{4.2}$$

The functions $\varepsilon_j^{\pm}(\lambda)$ have a simple physical meaning. The ground state of the model is characterised by the property that all the negative energy levels are filled, i.e. the Dirac sea is filled. The Dirac sea consists of the strings for which $\varepsilon_j^{-}(\lambda)\neq 0$. The energies of the hole type excitations are $\varepsilon_j^{-}(\lambda)$ and the energies of the particles are $\varepsilon_j^{+}(\lambda)$. The functions $\varepsilon_j(\lambda)$ may be calculated exactly at $H=0$. In this case they are either strictly positive or strictly negative, or identically zero in the whole range of variable λ. Below we shall consider only the case $H=0$.

The structure of the ground state of the system depends on the sign of the product $\varepsilon(-1)^r$.

(a) $\varepsilon=(-1)^r$. From the system (4.1) we conclude that in this case the Dirac sea consists of m_i strings with $i\equiv r+1 \pmod 2$, $i\leqslant r+1$. We now give an expression for the non-zero functions $\varepsilon_j(\lambda)$. It is most simple to write it down in terms of their Fourier transforms

$$\hat{\varepsilon}_j^{(0)}(x)=\hat{\varepsilon}_j(x)=4p_0\frac{\sinh[(j-m_i+1)p_{i+1}x]\cosh(q_\sigma x)}{\cosh(p_i x)\sinh(p_{i+1}x)} \qquad m_i\leqslant j\leqslant m_{i+1}-1 \tag{4.3}$$

$$\hat{\varepsilon}_{m_{i+1}}^{(0)}(x)=-\hat{\varepsilon}_{m_{i+1}}(x)=2p_0\frac{\cosh(q_0x)\sinh(b_ip_{i+1}x)}{\cosh(p_ix)\cosh(p_{i+2}x)\sinh(p_{i+1}x)} \qquad i\leqslant r-2 \tag{4.4}$$

$$\hat{\varepsilon}_j^{(0)}(x)=\hat{\varepsilon}_j(x)=4p_0\frac{\sinh[(j-m_r+1)p_{r+1}x]\cosh(q_\sigma x)}{\sinh(p_{r+1}x)\cosh(p_rx)} \qquad m_r\leqslant j\leqslant \sigma-1 \tag{4.5}$$

$$\hat{\varepsilon}_j^{(0)}(x)=\hat{\varepsilon}_j(x)=4p_0\frac{\sinh[(\sigma-m_r)p_{r+1}x]\cosh\{[q_j-(-1)^rp_{r+1}]x\}}{\sinh(p_{r+1}x)\cosh(p_rx)} \qquad \sigma-1\leqslant j\leqslant m_{r+1} \tag{4.6}$$

$$\hat{\varepsilon}_{m_{r+1}}^{(0)}(x)=-\hat{\varepsilon}_{m_{r+1}}(x)=2p_0\frac{\sinh[(\sigma-m_r)p_{r+1}x]}{\sinh(p_{r+1}x)\cosh(p_rx)}. \tag{4.7}$$

It is easy to calculate the asymptotics of the functions $\varepsilon_j(\lambda)$ at $\lambda\to\infty$

$$\begin{aligned}\varepsilon_j^{(0)}(\lambda)\simeq 4p_0\cos\left(\frac{\pi q_\sigma}{2p_i}\right)\sin\left((j-m_i+1)\frac{\pi p_{i+1}}{2p_i}\right)\\ \times\exp\left(-\frac{\pi\lambda}{2p_i}\right)\left[p_i\sin\left(\frac{\pi p_{i+1}}{2p_i}\right)\right]^{-1} \qquad m_i\leqslant j<m_{i+1}\end{aligned} \tag{4.8}$$

$$\varepsilon_{m_{i+1}}^{(0)}(\lambda)\simeq 2p_0\cos\left(\frac{\pi q_\sigma}{2p_i}\right)\exp\left(-\frac{\pi\lambda}{2p_i}\right)\left[p_i\sin\left(\frac{\pi p_{i+1}}{2p_i}\right)\right]^{-1} \qquad i\leqslant r,\ i=r \pmod 2. \tag{4.9}$$

These asymptotics are crucial to the investigation of the scaling limits of the anisotropic XXZ model.

(b) $\varepsilon=(-1)^{r+1}$. In this case the Dirac sea consists of the m_i strings with $i=r \pmod 2$, $i\leqslant r$ and also the $(\sigma-1)$ strings. The energies $\varepsilon_j^{\pm}(\lambda)$ with $j\leqslant m_r$ are given by (4.3) and (4.4). There is also an exact expression for $\varepsilon_{\sigma-1}^{-}(\lambda)$

$$\varepsilon_{\sigma-1}(\lambda)=-\varepsilon_{\sigma-1}^{-}(\lambda)=4p_0\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\lambda x)}{\cosh(p_{r+1}x)}\frac{\mathrm{d}x}{2\pi}=\frac{p_0}{p_{r+1}\cosh(\pi\lambda/2p_{r+1})}. \tag{4.10}$$

It is not difficult to calculate the momenta of the excitations

$$p_j^{(0)}(\lambda)=\int_{-\infty}^{\lambda}\varepsilon_j^{(0)}\,\mathrm{d}\mu. \tag{4.11}$$

From this formula one can obtain the excitation energies as functions of the physical momenta. Their behaviour at finite momenta remains obscure but for small momenta they are approximately linear. So we have the spectrum of sound type

$$\varepsilon_j^{(0)}(p)\simeq \pi p/2p_i \qquad p\to 0,\ m_{i-1}\leqslant j\leqslant m_i. \tag{4.12}$$

In the following we shall use the parameter λ instead of p. We call it the excitation rapidity.

It is easy to see that the functions (4.3) and (4.4) satisfy the following relations:

$$\varepsilon^{(0)}_{m_{i+1}}\{\lambda+\mathrm{i}[p_i-(j+1-m_i)p_{i+1}]\}+\varepsilon^{(0)}_{m_{i+1}}\{\lambda-\mathrm{i}[p_i-(j+1-m_i)p_{i+1}]\}=\varepsilon^{(0)}_j(\lambda) \tag{4.13}$$

$$\varepsilon^{(0)}_j[\lambda+\mathrm{i}(l-m_i+1)p_{i+1}]+\varepsilon^{(0)}_l[\lambda-\mathrm{i}(j-m_i+1)p_{i+1}]=\varepsilon^{(0)}_{[j+l-m_i+2]}(\lambda) \tag{4.14}$$

where $[j+l-m_i+2]=(j+l-2m_i-2)\bmod(b_i)+m_i$. These relations describe the bootstrap in the system. They imply that particle-like excitations (with $\varepsilon^+_j(\lambda)\neq 0$) may be interpreted as the bound states of the holes (i.e. the excitations with $\varepsilon^-_j(\lambda)\neq 0$). Moreover, the particle-like excitations form the bound states which are also particle-like. In analogy with the sine-Gordon model we may call the particle-like excitations 'breathers'. The hole-like excitations may be called kinks. Below we shall calculate the scattering amplitudes and we shall see that the poles of these amplitudes are in agreement with the shifts of arguments in (4.13) and (4.14).

Substituting the expressions (4.3)-(4.7) and (4.11) into (4.2) we obtain the ground-state energy at zero magnetic field:

$$\mathscr{E}_0(0)=-\sum_{\substack{i\leqslant r+1\\ i=r+1\ (\mathrm{mod}\,2)}}(-1)^{r+1}\int_{-\infty}^{+\infty}\hat{a}_{m_i,2S}(x)\hat{\varepsilon}^{(0)}_{m_i}(x)\,\mathrm{d}x \qquad \varepsilon=(-1)^r \tag{4.15}$$

$$\begin{aligned}\mathscr{E}_0(0)=-\sum_{\substack{i\leqslant r\\ i=r(\mathrm{mod}\,2)}}(-1)^{r}\int_{-\infty}^{+\infty}\hat{a}_{m_i,2S}(x)\hat{\varepsilon}^{(0)}_{m_i}(x)\,\mathrm{d}x\\ -(-1)^r\int_{-\infty}^{+\infty}\hat{a}_{\sigma-1,2S}(x)\hat{\varepsilon}^{(0)}_{\sigma-1}(x)\,\mathrm{d}x \qquad \varepsilon=(-1)^{r+1}.\end{aligned} \tag{4.16}$$

Let us introduce new notations for the string indices in accordance with the role of different strings in the composition of the ground state

$$\{j_0\}=\begin{cases}\{m_i|i\leqslant r,\ i=r\ (\mathrm{mod}\ 2)\}\cup\{\sigma-1\} & \varepsilon=(-1)^{r+1} \qquad (4.17)\\ \{m_i|i\leqslant r+1,\ i=r+1\ (\mathrm{mod}\ 2)\} & \varepsilon=(-1)^r \qquad (4.18)\end{cases}$$

$$\{j_1\}=\begin{cases}\{j\,|\,m_{i-1}\leqslant j<m_i,\ i\leqslant r,\ i=r\ (\mathrm{mod}\ 2)\} & \varepsilon=(-1)^{r+1} \qquad (4.19)\\ \{j\,|\,m_{i-1}\leqslant j<m_i,\ i\leqslant r+1,\ i=r+1\ (\mathrm{mod}\ 2)\} & \varepsilon=(-1)^r \qquad (4.20)\end{cases}$$

$$\{j_2\}=\{\text{other strings}\}. \tag{4.21}$$

The indices j_0 denote the sea strings. The j_1 strings correspond to the breathers and the j_2 strings have zero energy.

To calculate the scattering amplitudes let us transform the system (3.15) in the following way. Using equations (3.5) for the sea strings we express the densities ρ_{j_0} through the corresponding hole densities $\rho^{\mathrm{h}}_{j_0}$. Substituting these expressions for ρ_{j_0} into the equations for ρ_{j_1} and ρ_{j_2} we rewrite them in terms of hole densities of the sea strings. After some calculation we obtain the following system:

$$b_{k_0,\sigma}=\rho_{k_0}+\sum_{j_0}B_{k_0j_0}*\rho^{\mathrm{h}}_{k_0}+\sum_{k_1}A^{(0,1)}_{k_0k_1}*\rho_{k_1}+\sum_{k_2}A^{(0,2)}_{k_0k_2}*\rho_{k_2} \tag{4.22}$$

$$b_{k_1,\sigma}=\rho^{\mathrm{h}}_{k_1}+\sum_{j_1}A^{(1,1)}_{k_1j_1}*\rho_{j_1}+\sum_{k_0}A^{(1,0)}_{k_1k_0}*\rho^{\mathrm{h}}_{k_0}+\sum_{k_2}A^{(1,2)}_{k_1k_2}*\rho_{k_2} \tag{4.23}$$

$$\sum_{j_0}A^{(2,0)}_{k_2j_0}*\rho^{\mathrm{h}}_{j_0}+\sum_{k_1}A^{(2,1)}_{k_2k_1}*\rho_{k_1}=\varepsilon(-1)^{r(k_2)}\rho^{\mathrm{h}}_{k_2}+\sum_{j_2}A^{(2,2)}_{k_2j_2}*\rho_{j_2}. \tag{4.24}$$

The kernels $A^{(\alpha,\beta)}$, $B_{k_0j_0}$ and the functions $b_{k_0,\sigma}$, $b_{k_1,\sigma}$ are given in appendix 3.

Before the calculation of the scattering amplitudes let us find the spin of the excitations. Using the system (4.22)-(4.24) we can cast the expression (2.33) into the following form

$$S_z = z\{\tfrac{1}{2}\,\nu^{\mathrm{h}}_\alpha - M\}. \tag{4.25}$$

Here

$$z = (-1)^{r(\alpha)} p_0/q_\alpha$$

and $\alpha = m_{r+1}$ at $\varepsilon = (-1)^r$, $\alpha = \sigma - 1$ at $\varepsilon = (-1)^{r+1}$; $\nu^{\mathrm{h}}_\alpha = N\int_{-\infty}^{+\infty} \rho^{\mathrm{h}}_\alpha(\lambda)\,\mathrm{d}\lambda$ is the number of holes in α strings and $M = \Sigma_{k \geq \alpha+1} n^{(\tilde{p}_0)}_{k-\alpha}\nu_k$, where $n^{(\tilde{p}_0)}$ is the set of Takahaski numbers (3.7) defined by $\tilde{p}_0$. The values of $\tilde{p}_0$ are given by (A3.13) and (A3.14), $\nu_k = N\int_{-\infty}^{+\infty}\rho_k(\lambda)\,\mathrm{d}\lambda$.

Equation (4.25) implies that there is a finite spin renormalisation by the constant z. Renormalised spin of α kinks is equal to $\frac{1}{2}$. The spin of other kinks and the spin of the breathers is equal to zero.

There are many ways to calculate the scattering amplitudes (Korepin 1979, Destri and Lowenstein 1982, Woynarovich 1982, Tsvelick and Wiegmann 1983). We shall calculate them from Bethe equations over the physical vacuum (PBE) (Destri and Lowenstein 1982, Woynarovich 1982). This is a system of equations for the rapidities of the physical excitations which follows from the periodical boundary conditions for the wavefunction. The PBE system gives the equations (4.22)-(4.24) in the thermodynamical limit, when the number of physical excitations is macroscopically large.

We shall omit a derivation of the PBE, since it is rather lengthy. We present only the final result and demonstrate how to obtain the equations (4.22)-(4.24) from the BPE system in the thermodynamical region.

We shall write Bethe equations in terms of rapidities of physical excitations

$$\begin{aligned}
&\exp(\mathrm{i}p_{m_k}(\lambda^{(k)}_\alpha)N)\\
&\quad= \prod_{\tau=0,\pm 2}\prod_{\beta\neq\alpha}^{\nu_{k+\tau}} S_{m_k,m_{k+\tau}}(\lambda^{(k)}_\alpha - \lambda^{(k+\tau)}_\beta)\\
&\quad\times \prod_{\varepsilon=\pm 1}\prod_{j=1}^{b_{k+\varepsilon}}\prod_{\beta=1}^{\nu^{(j)}_k} S_{m_k,(k+\varepsilon,j)}(\lambda^{(k)}_\alpha - \Lambda^{(k+\varepsilon,j)}_\beta)\\
&\quad\times \prod_{\alpha=1}^{M_k}\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\lambda^{(k)}_\alpha - \mu^{(k)}_\beta + \mathrm{i}p_{k-1}(b_{k-2}-1)]\right)\\
&\quad\times\left[\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\lambda^{(k)}_\alpha - \mu^{(k)}_\beta - \mathrm{i}p_{k-1}(b_{k-2}-1)]\right)\right]^{-1}\\
&\quad\times \prod_{\alpha=1}^{M_{k+2}}\sinh\left(\frac{\pi}{2p_{k+1}b_k}(\lambda^{(k)}_\alpha - \mu^{(k+2)}_\beta + \mathrm{i}p_{k+1})\right)\\
&\quad\times\left[\sinh\left(\frac{\pi}{2p_{k+1}b_k}(\lambda^{(k)}_\alpha - \mu^{(k+2)}_\beta - \mathrm{i}p_{k+1})\right)\right]^{-1}
\end{aligned} \tag{4.26}$$

$$\begin{aligned}
&\exp(\mathrm{i}p_{j+m_k}(\Lambda^{(k,j)}_\alpha)N)\\
&\quad= \prod_{\varepsilon=\pm 1}\prod_{\beta=1}^{\nu^{\mathrm{h}}_{k+\varepsilon}} S_{(k,j),m_{k+\varepsilon}}(\Lambda^{(k,j)}_\alpha - \lambda^{(k+\tau)}_\beta)\prod_{l=1}^{b_k}\prod_{\beta=1}^{\nu^{(l)}_k} S_{(k,j)(k,l)}(\Lambda^{(k,j)}_\alpha - \Lambda^{(k,l)}_\beta)\\
&\quad\times\prod_{\varepsilon=\pm 1}\prod_{\alpha=1}^{M_k}\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\Lambda^{(k,j)}_\alpha - \mu^{(k)}_\beta\right.\\
&\qquad\left. + \mathrm{i}\varepsilon(p_{k-1} - jp_k) + \mathrm{i}p_{k-1}(b_{k-2}-1)]\right)
\end{aligned}$$

$$\times\left[\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\Lambda_\alpha^{(k,j)}-\mu_\beta^{(k)}+\mathrm{i}\varepsilon(p_{k-1}-jp_k)-\mathrm{i}p_{k-1}(b_{k-2}-1)]\right)\right]^{-1}. \tag{4.27}$$

Here $\lambda_\alpha^{(k)}$ are the rapidities of the holes in m_k strings and $\Lambda_\alpha^{(k,j)}$ are the rapidities of $(j+m_k)$ strings.

The functions $S_{kl}(\lambda)$, $S_{k,(l,j)}(\lambda)$ and $S_{(k,j)(l,j')}(\lambda)$ are defined by the following formulae

$$\begin{aligned}
S_{m_i m_k}(\lambda) &= \exp\left(\mathrm{i}\int_0^\lambda (B_{m_i m_k}(\mu)-\delta_{ik}\delta(\mu))\,\mathrm{d}\mu\right)\\
S_{m_k,(l,j)}(\lambda) &= \exp\left(\mathrm{i}\int_0^\lambda A^{(0,1)}_{m_{k,j}+m_{l-1}}(\mu)\,\mathrm{d}\mu\right) \qquad 1\leqslant j\leqslant b_l\\
S_{(k,j),(k,m)}(\lambda) &= \exp\left(\mathrm{i}\int_0^\lambda A^{(1,1)}_{j+m_k-1,m+m_k-1}(\mu)\,\mathrm{d}\mu\right) \qquad 1\leqslant j,m\leqslant b_k
\end{aligned} \tag{4.28}$$

or, in more detail,

$$S_{m_i m_i}(\lambda)=S\left(\frac{\lambda}{p_{i-1}};\,b_{i-2}\right)S\left(\frac{\lambda}{p_{i+1}};\,b_i\right)S\left(\frac{\lambda}{p_{i-1}};\,\frac{p_i}{p_{i-1}}\right)S\left(\frac{\lambda}{p_{i+1}};\,\frac{p_{i+2}}{p_{i+1}}\right)$$

$$\begin{aligned}
S(\lambda;\,p) &= \exp\left(⨍_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}x\lambda)}{x}\,\frac{\sinh[(p-1)x]}{2\cosh x\sinh(px)}\,\frac{\mathrm{d}x}{2\pi}\right)\\
&= \frac{\Gamma((-\mathrm{i}\lambda+2)/4)\Gamma(\mathrm{i}\lambda/4)\Gamma(-\mathrm{i}\lambda/2p)}{\Gamma((\mathrm{i}\lambda+2)/4)\Gamma(-\mathrm{i}\lambda/4)\Gamma(\mathrm{i}\lambda/2p)}\\
&\quad\times\prod_{m\geqslant 1}\frac{\Gamma^2((-\mathrm{i}\lambda+2mp+2)/4)\Gamma^2((\mathrm{i}\lambda+2mp)/4)}{\Gamma^2((\mathrm{i}\lambda+2mp+2)/4)\Gamma^2((-\mathrm{i}\lambda+2mp)/4)}
\end{aligned}$$

$$S_{m_i m_{i-2}(\lambda)}=S_{m_{i-2},m_i}(\lambda)=S\left(\frac{\lambda}{p_{i-1}};\,b_{i-2}\right)$$

$$S_{m_{r+1}m_{r+1}}(\lambda)=S\left(\frac{\lambda}{p_r};\,b_{r-1}\right)S\left(\frac{\lambda}{p_r};\,\frac{p_{r+1}}{p_r}\right)$$

$$S_{\sigma-1,\sigma-1}(\lambda)=S\left(\frac{\lambda}{p_{r+1}};\,\sigma-1-m_r\right)S\left(\frac{\lambda}{p_{r+1}};\,(-1)^r\frac{q_{\sigma-1}}{p_{r+1}}\right).$$

These functions satisfy the bootstrap relations

$$\begin{aligned}
S_{m_k,(l,j)}(\lambda) &= S_{m_k m_{l+1}}(\lambda+\mathrm{i}(p_l+jp_{l+1}))S_{m_k m_{l+1}}(\lambda-\mathrm{i}(p_l-jp_{l+1}))\\
S_{(l,m)(k,j)}(\lambda) &= S_{(l,m),m_{k+1}}(\lambda+\mathrm{i}(p_k-jp_{k+1}))S_{(l,m),m_{k+1}}(\lambda-\mathrm{i}(p_k-jp_{k+1}))
\end{aligned} \tag{4.13'}$$

$$S_{(k,j),(k,q)}(\lambda+\mathrm{i}lp_{k+1})S_{(k,l)(k,q)}(\lambda-\mathrm{i}jp_{k+1})=S_{(k,[j+l]),(k,q)}(\lambda) \tag{4.14'}$$

where $[j+l]=(j+l)\bmod b_k$.

If $\varepsilon=(-1)^{r+1}$, we have $k=r\ (\mathrm{mod}\ 2)$, $k<r$ in (4.26) and $k\leqslant r$ in (4.27). To obtain the equation for the rapidities of holes in m_r strings from (4.26) we must replace b_r

by $(\sigma-m_r-1)$ and $\lambda_\alpha^{(r+1)}$ by $\lambda_\alpha^{(\sigma-1)}$. The equation for the rapidities of holes in $(\sigma-1)$ strings has the following form

$$\begin{aligned}
&\exp[\mathrm{i}p_{\sigma-1}(\lambda_\alpha^{(\sigma-1)})N] \\
&\quad=\prod_{\beta=1}^{\nu_r^{\mathrm{h}}} S_{\sigma-1,r}(\lambda_\alpha^{(\sigma-1)}-\lambda_\beta^{(r)}) \prod_{\beta\neq\alpha}^{\nu_{\sigma-1}^{\mathrm{h}}} S_{\sigma-1,\sigma-1}(\lambda_\alpha^{(\sigma-1)}-\lambda_\beta^{(\sigma-1)}) \\
&\qquad\times\prod_{\beta=1}^{M_{\sigma-1}} \sinh\left(\frac{\pi}{2p_{r+1}(\sigma-m_r-1)}[\lambda_\alpha^{(\sigma-1)}-\mu_\beta^{(r+2)}+\mathrm{i}p_{r+1}(\sigma-m_r-2)]\right) \\
&\qquad\times\left[\sinh\left(\frac{\pi}{2p_{r+1}(\sigma-m_r-1)}[\lambda_\alpha^{(\sigma-1)}-\mu_\beta^{(r+2)}-\mathrm{i}p_{r+1}(\sigma-m_r-2)]\right)\right]^{-1} \\
&\qquad\times\prod_{\beta=1}^{M}\sinh\left(\frac{\pi}{2p_{r+1}p_0}(\lambda_\alpha^{(\sigma-1)}-\mu_\beta+\mathrm{i}p_{r+1})\right) \\
&\qquad\times\left[\sinh\left(\frac{\pi}{2p_{r+1}\tilde{p}_0}(\lambda_\alpha^{(\sigma-1)}-\mu_\beta-\mathrm{i}p_{r+1})\right)\right]^{-1}.
\end{aligned}\tag{4.29}$$

Here $\tilde{p}_0$ is defined by (A3.13). The numbers $\mu_\alpha^{(k)}$ parametrise the scattering states. Below we give a system of equations for these numbers.

At $\varepsilon=(-1)^r$ we have $k=r(\mathrm{mod}2)$, $k\leqslant r$ in (4.26) and $k\leqslant r+1$ in (4.27). Let $\tilde{p}_0$ be defined by (A3.14). Replacing b_{r+1} by $\tilde{p}_0$ we obtain an equation for the $\lambda_\alpha^{(r+1)}$.

The numbers $\mu_\alpha^{(k)}$ satisfy an auxiliary restriction:

$$|\mathrm{Im}\,\mu_\alpha^{(k)}|<\tfrac{1}{2}(b_{k-2}-1)$$

$$\begin{cases} k=r\ (\mathrm{mod}\ 2) & k\leqslant r,\ |\mathrm{Im}\,\mu_\alpha^{(\sigma-1)}|<\dfrac{\sigma-m_r-2}{2} \text{ at } \varepsilon=(-1)^{r+1} \\ k=r+1\ (\mathrm{mod}\ 2) & k\leqslant r+1 \text{ at } \varepsilon=(-1)^r \end{cases}\tag{4.30}$$

and they are the solutions of the following system:

$$\begin{aligned}
&\prod_{\beta=1}^{\nu_k^{\mathrm{h}}}\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\mu_\alpha^{(k)}-\lambda_\beta^{(k)}+\mathrm{i}p_{k-1}(b_{k-2}-1)]\right) \\
&\qquad\times\left[\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\mu_\alpha^{(k)}-\lambda_\beta^{(k)}-\mathrm{i}p_{k-1}(b_{k-2}-1)]\right)\right]^{-1} \\
&\qquad\times\prod_{\beta=1}^{\nu_{k-2}^{\mathrm{h}}}\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}(\mu_\alpha^{(k)}-\lambda_\beta^{(k-2)}+\mathrm{i}p_{k-1})\right) \\
&\qquad\times\left[\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}(\mu_\alpha^{(k)}-\lambda_\beta^{(k-2)}-\mathrm{i}p_{k-1})\right)\right]^{-1} \\
&\qquad\times\prod_{\varepsilon=\pm1}\prod_{j=1}^{b_{k-1}}\prod_{\beta=1}^{\nu_k^{(j)}}\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\mu_\alpha^{(k)}-\Lambda_\beta^{(k,j)}\right. \\
&\qquad\left.+\mathrm{i}\varepsilon(p_{k-1}-jp_k)+\mathrm{i}p_{k-1}(b_{k-2}-1)]\right) \\
&\qquad\times\left[\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}[\mu_\alpha^{(k)}-\Lambda_\beta^{(k,j)}+\mathrm{i}\varepsilon(p_{k-1}-jp_k)-\mathrm{i}p_{k-1}(b_{k-2}-1)]\right)\right]^{-1} \\
&\quad=\prod_{\beta\neq\alpha}^{M_k}\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}(\mu_\alpha^{(k)}-\mu_\beta^{(k)}+2\mathrm{i}p_{k-1})\right) \\
&\qquad\times\left[\sinh\left(\frac{\pi}{2p_{k-1}b_{k-2}}(\mu_\alpha^{(k)}-\mu_\beta^{(k)}-2\mathrm{i}p_{k-1})\right)\right]^{-1}.
\end{aligned}\tag{4.31}$$

The equation for the numbers $\mu_\alpha^{(r+1)}$ in the case $\varepsilon=(-1)^{r+1}$ are the same but instead of $\lambda_\alpha^{(r+2)}$ and b_r we must write $\lambda_\alpha^{(\sigma-1)}$ and $(\sigma-m_r+1)$, respectively, and the multiplier with numbers $\Lambda_\alpha^{(r+2,j)}$ must be omitted.

The equations for the numbers μ_α at $\varepsilon=(-1)^{r+1}$ have the following form:

$$\begin{aligned}\prod_{\beta=1}^{\nu^{\mathrm{h}}_{\sigma-1}} &\sinh\left(\frac{\pi}{2\tilde{p}_0 p_{r+1}}(\mu_\alpha-\lambda_\beta^{(\sigma-1)}+\mathrm{i}p_{r+1})\right)\\ &\times\left[\sinh\left(\frac{\pi}{2\tilde{p}_0 p_{r+1}}(\mu_\alpha-\lambda_\beta^{(\sigma-1)}-\mathrm{i}p_{r+1})\right)\right]^{-1}\\ &=\prod_{\beta\neq\alpha}^{M}\sinh\left(\frac{\pi}{2\tilde{p}_0 p_{r+1}}(\mu_\alpha-\mu_\beta+2\mathrm{i}p_{r+1})\right)\\ &\times\left[\sinh\left(\frac{\pi}{2\tilde{p}_0 p_{r+1}}(\mu_\alpha-\mu_\beta-2\mathrm{i}p_{r+1})\right)\right]^{-1}.\end{aligned}\tag{4.32}$$

At $\varepsilon=(-1)^r$ the system is the same but the numbers $\lambda_\alpha^{(\sigma-1)}$ and p_{r+1} must be replaced by the numbers $\lambda_\alpha^{(r+1)}$ and p_{r+2}, respectively.

The thermodynamical limit over the ground state corresponds to a macroscopically large amount of excitations. In this limit the numbers $\lambda_\alpha^{(k)}$, $\Lambda_\alpha^{(k,j)}$ and $\mu_\alpha^{(k)}$ fill in the real axis with densities ρ_{j_0}, ρ_{j_1} and ρ_{j_2} and equations (4.26)-(4.32) give the system (4.22)-(4.24), namely the numbers $\lambda_\alpha^{(k)}$ and $\Lambda_\alpha^{(k,j)}$ fill in the real axis with densities $\rho^{\mathrm{h}}_{m_k}(\lambda)$ and $\rho_{j+m_k-1}(\lambda)$ correspondingly. The numbers $\mu_\alpha^{(k)}$ at $\nu_k^{\mathrm{h}}\gg 1$ form strings with length $1\leqslant j\leqslant b_{k-2}-1$ and positive parities. Strings with length 1 and negative parity are forbidden by the constraint (4.30). The j strings in $\mu_\alpha^{(k)}$ at $\nu_k^{\mathrm{h}}\gg 1$ fill in the real axis with the densities $\rho_{j+m_k-1}(\lambda)$. In a similar way the numbers μ_α also form strings. These strings fill in the real axis with the densities $\rho_{j_2}(\lambda)$, where $j_2>\sigma-1$ at $\varepsilon=(-1)^{r+1}$ and $j_2>m_{r+1}$ at $\varepsilon=(-1)^r$.

Equations (4.26)-(4.32) contain all information about scattering amplitudes. The right-hand sides of equations (4.26), (4.27) and (4.29) are equal to the eigenvalues of the full S matrix (Destri and Lowenstein 1982). The equations (4.31) and (4.32) are called the high level Bethe ansatz equations. The solutions of these equations parametrise the eigenvectors and eigenvalues of the full S matrix. Comparing these equations with the known Bethe ansatz formulae we restore the two-particle S matrices.

If we ignore the restrictions (4.30) we obtain the following kink-kink S matrices

$$\begin{aligned}\hat{\mathbb{S}}^{(k,k)}(\lambda)&=S_{m_k m_k}(\lambda)1\otimes\ldots\otimes\hat{S}^{(b_{k-2}-1,b_{k-2}-1)}\left(\frac{\lambda}{p_{k+1}};\frac{\pi}{b_{k-2}}\right)\otimes\hat{S}^{(1,1)}\left(\frac{\lambda}{p_{k+1}};\frac{\pi}{b_k}\right)\otimes\ldots\otimes 1\\ \hat{\mathbb{S}}^{(k,k-2)}(\lambda)&=S_{m_k m_{k-2}}(\lambda)1\otimes\ldots\otimes\hat{S}^{(b_{k-2}-1,1)}\left(\frac{\lambda}{p_{k-1}};\frac{\pi}{b_{k-2}}\right)\otimes\ldots\otimes 1\end{aligned}\tag{4.33}$$

$$\begin{aligned}\hat{\mathbb{S}}^{(r+1,r+1)}(\lambda)&=S_{m_{r+1},m_{r+1}}(\lambda)1\otimes\ldots\otimes\hat{S}^{(b_{r-1}-1,b_{r-1}-1)}\left(\frac{\lambda}{p_r};\frac{\pi}{b_{r-1}}\right)\\ &\quad\otimes\hat{S}^{(1,1)}\left(\frac{\lambda}{p_{r+2}};\frac{\pi}{\tilde{p}_0}\right)\qquad\varepsilon=(-1)^r\\ \hat{\mathbb{S}}^{(\sigma-1,\sigma-1)}(\lambda)&=S_{\sigma-1,\sigma-1}(\lambda)1\otimes\ldots\otimes\hat{S}^{(\sigma-m_r-2,\sigma-m_r-2)}\left(\frac{\lambda}{p_{r+1}};\frac{\pi}{\sigma-m_r-1}\right)\\ &\quad\otimes\hat{S}^{(1,1)}\left(\frac{\lambda}{p_{r+1}};\frac{\pi}{\tilde{p}_0}\right)\qquad\varepsilon=(-1)^{r+1}.\end{aligned}\tag{4.34}$$

Here matrices $\hat{S}^{(l,m)}(\lambda, p)$ are connected with the matrices (2.3) by the following relation

$$\hat{S}^{(l,m)}(\lambda; p) = \prod_{j=0}^{m-1} \frac{1}{\sinh\{u + i\gamma[(m+l)/2 - j]\}} R^{(l,m)}(u - i\gamma(m+l-2)/2) \qquad l \geqslant m \tag{4.35}$$

where $u = \pi\lambda/2p$, $\gamma = -\pi/p$.

In accordance with the bootstrap formulae (4.13), (4.14) and (4.13′) and (4.14′) the breather-breather and the kink-breather S matrices can be obtained as the bound state S matrices (see Karowski 1979). Many-particle S matrices are equal to the product of the two-particle ones.

The structure of two-particle S matrices shows that the excitations in our model have auxiliary 'hidden' degrees of freedom. We call them 'hidden spins'. From (4.34)-(4.37) we conclude that for the kth kink the $(k+1)$th hidden spin variable takes the value $\frac{1}{2}$ while the value of the kth hidden spin variable is equal to $(b_{k-2}-1)/2$. The kinks which correspond to 'lower Dirac sea' (the holes in m_{r+1} strings at $\varepsilon = (-1)^r$ and the holes in $(\sigma - 1)$ strings at $\varepsilon = (-1)^{r+1}$) have the renormalised original spin $\frac{1}{2}$ and rth hidden spin variables take the value $(b_{r-1}-1)/2$ at $\varepsilon = (-1)^r$ and $(\sigma - m_r - 2)/2$ at $\varepsilon = (-1)^{r+1}$. The other hidden spins of these kinks are equal to zero.

In the case $S = \frac{1}{2}$ there are no 'hidden spins'. A systematic analysis of the Bethe equation in this case was given by Babelon *et al* (1983).

Let us return now to the restrictions (4.30). They are the constraints on the scattering states and they admit only a part of the eigenstates of the S matrices described above. As a consequence, the number of degrees of freedom in the n-particle sector is much less than the product of dimensions of the one-particle spaces. This phenomenon appears also for the isotropic higher-spin Heisenberg model (Andrei and Destri 1983, Faddeev and Reshetikhin 1984, 1986). Its interpretation is not known at present.

To conclude let us point out that the constraints (4.30) do not play any role in the limit $S \to \infty$ and thus pose no problems for the description of scattering states in this limit. This fact is useful for the solution of the quantum anisotropic principal chiral field model and O(3) non-linear σ model (Wiegmann 1984, Faddeev and Reshetikhin 1986).

5. Conclusion

We have given here a detailed description of the *XXZ* model with higher spin and derived the equation which governs its thermodynamic behaviour. Our argument relied essentially on the assumption that the spin and the anisotropy parameter are commensurable. This assumption emerges from the analysis of the string solution to the Bethe equations. It ensures that there are no restrictions on the positions of the centrum of admissible strings. Conversely, if it fails, there are always some non-trivial restrictions on these positions. In the thermodynamical limit the range of the admissible values of $\lambda^{(j)}$ depends on densities ν_j/N where ν_j is the total value of j strings in a given state. In some cases the constraints for the positions of the strings' centres appear already over the ferromagnetic vacuum.

A correlation between the spin and the anisotropy parameter is, of course, by no means casual. As a matter of fact, operators S_n^3 are present in the Hamiltonian only in the form of algebraic combinations of $\exp(i\gamma S_n^3)$. Hence the model is characterised by two periods. The first one is connected with rotations in U(1) and is related to the spin value. The second one is connected with the Hamiltonian and is equal to $p_0 = \pi/\gamma$. Obviously, if $S < p_0$ the values of $\exp(\mathrm{i}\gamma S^3)$ for $-S \leqslant S^3 \leqslant S$ sweep a part of the unit circle only once. The situation does not differ from the rational case (corresponding to $\gamma \to 0$). On the other hand, if $S > p_0$ then $\exp(\mathrm{i}\gamma S^3)$ winds over the circle several times. As we conclude from the exact solution, the structure of the ground state is then completely different. Thus our model gives an example of a quantum integrable system for which the commensurability effects are present.

Another interesting feature of our model is the existence of hidden spin variables. This phenomenon was found for the isotropic model with higher spin as well. At present it does not have any qualitative explanations.

The case of rational p_0 and $1+2S = y_{\alpha+1} = n_{m_{\alpha+1}}$ corresponds to the lattice sine-Gordon model considered by Bogolubov and Izergin (1984).

It is especially interesting to consider the magnetic anisotropy in the limit $S \to \infty$. This limiting case corresponds to a relativistic quantum field model, its scattering matrix being the tensor product of the kink S matrices for the Gross–Neveu and sine-Gordon models. It is tentative to conjecture that this scattering matrix corresponds to the anysotropic SU(2) chiral model (Wiegmann 1984).

After this work was completed we have learned that the case $S < \pi/\gamma$ had been studied by Babujian and Tsvelick (1986).

Acknowledgment

The authors are grateful to L Faddeev, N Bogolubov, A Izergin, V Korepin, E Sklyanin and F Smirnov for valuable and stimulating discussions and to M Semenov-Tian-Shanski for improvements in the English text. One of us (NR) wishes to thank A Tsvelick for useful discussions.

Appendix 1

Below we give some properties of the functions $\hat{a}_j(x)$, $\hat{n}_j(x)$, $\hat{a}_{j,2S}(x)$ defined by the formulae (3.18), (3.17) and (3.21). All calculation in the main text are based on these properties.

The functions $\hat{n}_j(x)$ and $\hat{a}_j(x)$ satisfy the following relations

(1)

$$\hat{n}_j(x) = \hat{S}_i(x)(\hat{n}_{j-1}(x) + \hat{n}_{j+1}(x)) \qquad m_{i-1} < j < m_i - 1 \tag{A1.1}$$

$$\hat{n}_{m_i-1}(x) = \hat{d}_i(x)\hat{n}_{m-1}(x) + (1-2\delta_{m_i m_{i-1}-1})\hat{n}_{m_i-2}(x)\hat{S}_i(x) + \hat{n}_{m_i}(x)\hat{S}_{i+1}(x) \tag{A1.2}$$

$$\hat{n}_{m_i}(x) = \hat{S}_{i+1}(x)(-\hat{n}_{m_i-1}(x) + \hat{n}_{m_i+1}(x)). \tag{A1.3}$$

Here $\hat{S}_i(x)$ is defined by (2.22).

(2)

$$\hat{a}_j(x)\hat{n}_k(x) - \hat{a}_k(x)\hat{n}_j(x) = (-1)^i \hat{n}_{\tilde{k}}^{(\tilde{p}_0)}(p_{i+1}x) \tag{A1.4}$$

where $\tilde{p}_0 = (-1)^i q_j/p_{i+1}$, $\tilde{k} = k - j$ and $\hat{n}_j^{(\tilde{p}_0)}(x)$ are the functions $\hat{n}_j(x)$ (see (3.7)) which are defined by the number $\tilde{p}_0$.

(3)

$$\hat{n}_k(x) = \hat{n}_{k_0}(x)\hat{A}_{k_0 k}^{(0,\alpha)}(x) \qquad \hat{a}_k(x) = \hat{a}_{k_0}(x)\hat{A}_{k_0 k}^{(0,\alpha)}(x). \tag{A1.5}$$

Here $\alpha=1,2$ and $k\in\{j_\alpha\}$ (see (4.17)-(4.21)). For the first j the functions $\hat{n}_j(x)$ are given by the following simple formulae

$$\hat{n}_j(x)=\coth x\sinh jx \qquad 1\leqslant j<m_1$$
$$\hat{n}_{m_1}(x)=\cosh[(p_0-1)x] \qquad \hat{n}_{m_2}(x)=\coth x\sinh(b_0x).$$

The functions $\hat{a}_{j,2S}(x)$ as well as $\hat{A}_{jk}(x)$ can be expressed through the functions $\hat{a}_j(x)$ and $n_j(x)$

$$\hat{a}_{j,2S}(x)=\hat{a}_j(x)\left\{\hat{n}_\sigma(x)-\cosh\left[\left(\left\{\frac{n_\sigma}{p_0}\right\}-\frac{1-(-1)^r}{2}\right)p_0x\right]\right\} \qquad n_j\geqslant 2S$$

$$\hat{a}_{j,2S}(x)=\hat{a}_\sigma(x)\hat{n}_j(x)+\cosh(q_\sigma x)\frac{\sinh\{[-q_j+(-1)^{r(j)}p_0]x\}}{\sinh(p_0x)} \qquad n_j\leqslant 2S.$$

Appendix 2

Let us give a general method for inverting the matrices which we use in this paper.

Let $x_1\ldots x_n$, $y_1\ldots y_n$ be two sets of the variables and $x_0=1$, $y_0=0$, $x_{n+1}=0$, $y_{n+1}=1$.

Theorem. If the matrix $A=(a_{ij})$ has the form

$$a_{ij}=\begin{cases}x_iy_j & i\geqslant j\\ x_jy_i & i\leqslant j\end{cases}$$

and $B=(b_{ij})$ is the inverse matrix, then

$$\begin{aligned}&b_{ii}=(x_{i-1}y_{i+1}-x_{i+1}y_{i-1})b_{i-1,i}b_{i,i+1} && 1\leqslant i\leqslant n\\ &b_{i-1,i}=-(x_{i-1}y_i-x_iy_{i-1})^{-1} && 2\leqslant i\leqslant n\end{aligned} \tag{A2.1}$$

$$\begin{aligned}&b_{i,i+1}=-(x_iy_{i+1}-x_{i+1}y_i)^{-1} && 1\leqslant i\leqslant n-1\\ &b_{ij}=0 && |i-j|\geqslant 2.\end{aligned} \tag{A2.2}$$

Using this theorem we compute the inverse matrix of $A_{j_0k_0}(-1)^{r(k_0)}$. If $\varepsilon=(-1)^{r+1}$ this inverse matrix has the form

$$\hat{B}_{m_{i-2}m_i}(x)=\frac{\sinh(p_{i-1}x)}{2\sinh(p_{i-1}b_{i-2}x)\cosh(p_{i-1}x)} \tag{A2.3}$$

$$\hat{B}_{m_im_{i+2}}(x)=\frac{\sinh(p_{i+1}x)}{2\sinh(p_{i+1}b_ix)\cosh(p_{i+1}x)} \tag{A2.4}$$

$$\hat{B}_{m_im_i}(x)=\frac{\sinh(p_{i-2}x)\sinh(p_{i-1}x)}{2\sinh(p_ix)\sinh(p_{i-1}b_{i-2}x)\cosh(p_{i-1}x)}+\frac{\sinh(p_{i+2}x)\sinh(p_{i+1}x)}{2\sinh(p_ix)\sinh(p_{i+1}b_ix)\cosh(p_{i+1}x)}. \tag{A2.5}$$

Here $i\equiv r\ (\mathrm{mod}\ 2)$, $i<r$

$$\hat{B}_{m_r\sigma-1(x)}=\frac{\sinh(p_{r+1}x)}{2\sinh[(\sigma-m_r-1)p_{r+1}x]\cosh(p_{r+1}x)} \tag{A2.6}$$

$$\hat{B}_{m_r m_r}(x)=\frac{\sinh(p_{r-2}x)\sinh(p_{r-1}x)}{2\sinh(p_r x)\sinh(p_{r-1}b_{r-2}x)\cosh(p_{r-1}x)}$$
$$+\frac{\sinh(p_{r+2}x)\sinh(p_{r+1}x)}{2\sinh(p_r-x)\sinh[p_{r+1}(\sigma-m_r-1)x]\cosh(p_{r+1}x)} \tag{A2.7}$$

$$\hat{B}_{\sigma-1,\sigma-1}(x)=(-1)^{r+1}\frac{\sinh(p_{r+1}x)\sinh(p_r x)}{2\sinh(q_{\sigma-1}x)\sinh[p_{r+1}(\sigma-m_r-1)x]\cosh(p_{r+1}x)}. \tag{A2.8}$$

If $\varepsilon=(-1)^r$ the matrix elements of $B_{k_0 j_0}$ are given by the formulae (A2.3)-(A2.5) if $i\neq r+1$ and

$$\hat{B}_{m_{r+1}m_{r+1}}(x)=\frac{\sinh(p_r x)\sinh(p_{r-1}x)}{2\sinh(p_{r+1}x)\sinh(p_r b_{r-1}x)\cosh(p_r x)}. \tag{A2.9}$$

Appendix 3

The functions $b_{k_0,\sigma}$ and $b_{k_1,\sigma}$ coincide with the energies of corresponding strings:

$$b_{k_0,\sigma}(\lambda)=\varepsilon^{(0)}_{k_0}(\lambda)\qquad b_{k_1,\sigma}(\lambda)=\varepsilon^{(0)}_{k_1}(\lambda). \tag{A3.1}$$

When $\varepsilon=(-1)^{r+1}$ we write for the kernels $A^{(0,\alpha)}_{k_0 j\alpha}$:

$$\hat{A}^{(0,1)}_{m_i k}(x)=\begin{cases}\dfrac{\sinh(q_k x)}{\sinh(q_{m_i}x)}+\dfrac{\sinh(p_{i-1}x)\sinh(p_{i-2}x)\cosh[(q_k+q_{m_i})x]}{\cosh(p_{i-1}x)\sinh(p_i x)\sinh(p_{i-1}b_{i-2}x)} & \\ \qquad\qquad i=r(\mathrm{mod}\ 2),\ i\leqslant r,\ m_{i-1}\leqslant k<m_i & \\ -\dfrac{\sinh(p_{i+1}x)\cosh[(q_k+q_{m_{i+2}})x]}{\cosh(p_{i+1}x)\sinh(p_{i+1}b_i x)} & \\ \qquad\qquad i=r(\mathrm{mod}\ 2),\ i<r,\ m_{i+1}\leqslant k<m_{i+2} & \\ 0 \qquad \text{in other cases} & \end{cases} \tag{A3.2}$$

$$\hat{A}^{(0,1)}_{\sigma-1,k}(x)=0 \tag{A3.3}$$

$$\hat{A}^{(0,2)}_{m_i k}(x)=\begin{cases}\dfrac{\sinh[p_{i-1}(k-m_{i-2})x]}{\sinh(p_{i-1}b_{i-2}x)} & m_{i-2}<k<m_{i-1},\ i=r\ (\mathrm{mod}\ 2),\ i<r\\ \dfrac{\sinh[p_{i+1}(m_{i+1}-k)x]}{\sinh(p_{i+1}b_i x)} & m_i<k<m_{i+1},\ i=r\ (\mathrm{mod}\ 2),\ i<r\\ 0 & \text{in other cases}\end{cases} \tag{A3.4}$$

$$\hat{A}^{(0,2)}_{m_r,k}(x)=\begin{cases}\dfrac{\sinh[p_{r-1}(k-m_{r-2})x]}{\sinh(p_{r-1}b_{r-2}x)} & m_{r-2}<k<m_{r-1}\\ \dfrac{\sinh[(\sigma-1-k)p_{r+1}x]}{\sinh[(\sigma-1-m_r)p_{r+1}x]} & m_r<k<\sigma-1\\ 0 & \text{in other cases}\end{cases} \tag{A3.5}$$

$$\hat{A}^{(0,2)}_{\sigma-1,k}(x)=\begin{cases}\dfrac{\sinh[p_{r+1}(k-m_r)x]}{\sinh[p_{r+1}(\sigma-1-m_r)x]} & m_r<k<\sigma-1\\ \dfrac{\sinh(q_kx)}{\sinh(q_{\sigma-1}x)} & k>\sigma-1\\ 0 & \text{in other cases.}\end{cases}\tag{A3.6}$$

When $\varepsilon=(-1)^r$ the functions $\hat{A}^{(0,1)}_{m_ik}$ are defined by (A3.2) but now $i=r+1 \pmod 2)$, $i<r+1$. The matrix $\hat{A}^{(0,2)}_{m_ik}$ at $i\leqslant r$ is defined by (A3.4) and for $\hat{A}^{(0,2)}_{m_{r+1},k}$ we have

$$\hat{A}^{(0,2)}_{m_{r+1},k}(x)=\begin{cases}0 & k\leqslant m_{r-1}\\ \dfrac{\sinh(p_r(k-m_{r-1})x)}{\sinh(p_rb_{r-1}x)} & m_{r-1}\leqslant k<m_r\\ \dfrac{\sinh(q_kx)}{\sinh(q_{m_r}x)} & k\geqslant m_{r+1}.\end{cases}\tag{A3.7}$$

The matrix $\hat{B}_{k_0j_0}$ is the inverse of $\hat{A}_{k_0j_0}(-1)^{r(j_0)}$ and is given in appendix 2.
The matrix $\hat{A}^{(1,\alpha)}$ has the form

$$\hat{A}^{(1,1)}_{jk}(x)=\begin{cases}2\cosh((q_k+q_{m_i})x)\hat{A}^{(0,1)}_{m_i,j}(x) & m_{i-1}<j\leqslant k<m_i\\ 0 & \text{in other cases}\end{cases}\tag{A3.8}$$

$$\hat{A}^{(1,0)}_{j_1k_0}(x)=\hat{A}^{(0,1)}_{k_0j_1}(x)\tag{A3.9}$$

$$\hat{A}^{(1,2)}_{jk}(x)=\begin{cases}2\cosh\{[q_j+(-1)^{r(j)}p_{i+1}]x\}\dfrac{\sinh(p_i(k-m_{i-1})x)}{\sinh(p_ib_{i-1}x)} & \begin{array}{l}m_{i-1}<k<m_i\\ m_i\leqslant j<m_{i+1}\end{array}\\ 0 & \text{in other cases.}\end{cases}\tag{A3.10}$$

If $\varepsilon=(-1)^{r+1}$ we have for the matrix $\hat{A}^{(2,1)}$ the following expressions:

$$\hat{A}^{(2,2)}_{jk}(x)=\hat{A}^{(b_{i-1})}_{\tilde{j}\tilde{k}}(p_ix)\qquad \begin{array}{l}m_{i-1}<j,k<m_i\\ \tilde{j}=j-m_{i-1},\ \tilde{k}=k-m_{i-1}\end{array}\tag{A3.11}$$

$$\hat{A}^{(2,2)}_{jk}(x)=\hat{A}^{(\sigma-m_r-1)}_{\tilde{j}\tilde{k}}(p_{r+1}x)\qquad \begin{array}{l}m_r<j,\ k<\sigma-1\\ \tilde{j}=j-m_r,\ \tilde{k}=k-m_r\end{array}\tag{A3.12}$$

$$\hat{A}^{(2,2)}_{jk}(x)=\hat{A}^{(\tilde{p}_0)}_{\tilde{j}\tilde{k}}(p_{r+1}x)\qquad \begin{array}{l}j,\ k>\sigma-1\\ \tilde{j}=j-\sigma+1,\ \tilde{k}=k-\sigma+1\end{array}\qquad \tilde{p}_0=(-1)^r\frac{q_{\sigma-1}}{p_{r+1}}.\tag{A3.13}$$

If $\varepsilon=(-1)^r$ the matrix $\hat{A}^{(2,2)}_{jk}(x)$ at $j,\ k<m_{r+1}$ is defined by (A3.11). At $j,\ k>m_{r+1}$ this matrix is the same as (A3.13):

$$\hat{A}^{(2,2)}_{jk}(x)=\hat{A}^{(\tilde{p}_0)}_{\tilde{j}\tilde{k}}(x)\qquad \tilde{j}=j-m_{r+1}\qquad \tilde{k}=k-m_{r+1}\qquad \tilde{p}_0=\frac{p_{r+1}}{p_{r+2}}\tag{A3.14}$$

$$\hat{A}^{(2,1)}_{j_2k_1}(x)=\hat{A}^{(1,2)}_{k_1j_2}(x)\qquad \hat{A}^{(2,0)}_{j_2k_0}(x)=\hat{A}^{(0,2)}_{k_0j_2}(x)\tag{A3.15}$$

All the formulae from this appendix are proved in Kirillov and Reshetikhin (1985b).

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